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SPECIFIC DAMPING CAPACITY FOR ARBITRARY LOSS ANGLE

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An analysis was performed to establish a relationship between specific damping capacity and loss angle for materials with arbitrary loss angle. The motivation for this work is that the usual relationship is only valid for low loss materials and leads to some confusion when the specific damping capacity is greater than one. In this paper, two equations for specific damping capacity are derived that are valid for all values of loss angle. One equation uses the usual definition of specific damping capacity as dissipated energy per cycle divided by maximum stored energy. The other equation defines specific damping capacity as dissipated energy per cycle divided by work done per cycle. The latter equation has the desirable property of varying between zero and one. The analysis done in the time domain is extended to include analysis of hysteresis curves as well.

1. INTRODUCTION

The purpose of this paper is to establish the relation between specific damping capacity and loss angle for materials of arbitrary loss angle. The motivation for this work is that the usual relation between these variables is only valid for low loss materials [1–10]. Because the exact relation is unavailable, the range of validity of the low loss approximation is not known. Having gone through the analysis to determine the relation for arbitrary loss angle, it becomes apparent that the definition for specific damping capacity is not the most useful and an alternate definition is proposed. The analysis carried out in the time domain is then applied to the evaluation of stress–strain hysteresis loops.

A common metric of energy loss in a dynamic mechanical experiment is the loss factor, which is defined as $\tan\delta$, where δ is the loss angle by which strain lags stress. This definition is based on viewing stress and strain as complex variables, leading, for example, to a complex shear modulus, $\mathbf{G}^* = G' + \mathrm{i}G''$, where the angle in the complex plane between loss modulus, G'', and storage modulus, G', is δ so that $\tan\delta = G''/G'$. Experimental measurements in the time domain generally yield the loss angle (or loss factor). The presentation here will specifically consider only shear deformation though the extension to other modes of deformation is straightforward.

Another metric of energy loss is the specific damping capacity, $\Delta W/W$, which is not measured directly but determined through a relationship to loss angle. ΔW is the total energy dissipated per cycle. The meaning and intention of W is not so clear. The intention is to evaluate the maximum stored (elastic) energy but various approximations are used. All of the approximations assume that the loss angle is small. The most common approximation is the one given, for example, by Read and Dean [4],

$$\Delta W/W_s \approx 2\pi \tan\delta, \qquad \delta \ll 1, \tag{1}$$

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where the subscript on W will be used here to distinguish this stored energy definition with another definition proposed later. Zener [1] used a slightly different approximation and obtained

$$\Delta W/W_s \approx 2\pi \sin \delta, \qquad \delta \ll 1. \tag{2}$$

For small δ, equations (1) and (2) are nearly identical.

One purpose of this paper is to eliminate the restriction of small loss angle from equations (1) and (2) by defining specific damping capacity for arbitrary δ. A second purpose of this paper is to propose a new definition of specific damping capacity in terms of the work done per cycle (W_d) rather than the maximum work stored. As described below, this new definition is more intuitively obvious and has the advantage that the quantity $\Delta W/W_d$ is a fraction that can never be greater than one.

2. SPECIFIC DAMPING CAPACITY FOR ARBITRARY δ

Consider a linear material subjected to low amplitude, time harmonic shear stress of maximum amplitude σ_0 and frequency ω. The steady state stress has the form

$$\sigma = \sigma_0 \sin(\omega t) \tag{3}$$

Some of the stressing energy is dissipated in the form of heat. The resulting shear strain lags behind the stress by a phase angle δ such that

$$\gamma = \gamma_0 \sin(\omega t - \delta), \tag{4}$$

where γ_0 is the maximum shear strain amplitude. The phase angle δ is known as the shear loss angle. For an ideal elastic material, no energy is converted to heat and $\delta = 0$. The larger δ becomes, the greater is the amount of energy converted to heat. For a purely viscous material, all the energy is converted to heat and $\delta = \pi/2$ [11].

Work per unit volume can be calculated from the integral of the instantaneous stress times the infinitesimal strain

$$W = \int_a^b \sigma \, \mathrm{d}\gamma, \tag{5}$$

where the limits of equation (5) depend on the particular problem. When stress and strain are harmonic functions, it is convenient to express W as an integral in the time domain

$$W = \int_a^b \sigma(t) \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} \, \mathrm{d}t, \tag{6}$$

where the integrand ($\mathrm{d}W/\mathrm{d}t = \sigma(t)\, \mathrm{d}\gamma(t)/\mathrm{d}t$) is the instantaneous rate of doing work. Substituting equations (3) and (4) into equation (6) yields

$$W = \frac{\sigma_0 \gamma_0 \omega}{2} \int_a^b [\cos \delta \sin(2\omega t) - \sin \delta \cos(2\omega t) + \sin \delta] \, \mathrm{d}t. \tag{7}$$

It is instructive to plot the integrand of equation (7), instantaneous work, along with the associated stress and strain (equations (3) and (4)) as functions of time. The steady state results for a typical cycle are shown in Figure 1 for the case of a low loss material, $\delta = 0{\cdot}05$. As required by equation (7), $\mathrm{d}W/\mathrm{d}t$ oscillates at twice the frequency of the stress.

The dW/dt curve is initially positive and increases with time. The instantaneous work increases to a maximum and then goes to zero when the strain is a maximum. This occurs at a time $t = (\pi/2 + \delta)/\omega$. Note also that this time is not exactly a quarter cycle ($t = \pi/2\omega$) as usually assumed. Also, recall that the stress and strain do not reach a maximum at the same time (except for $\delta = 0$). As time progresses, dW/dt becomes negative. The dW/dt curve decreases to a minimum and then goes to zero at $t = \pi/\omega$. The stress is zero at this time; the strain lags behind by δ. In the second half cycle, the shape of the dW/dt curve is like the first half cycle, but the stress and strain curves are negative. The instantaneous work is zero when the strain is a minimum, at $t = (3\pi/2 + \delta)/\omega$, and is again zero when the stress is zero, at $t = 2\pi/\omega$.

The areas under the dW/dt curve represents work. To determine the work, equation (7) is integrated, yielding

$$W = (\sigma_0\gamma_0/2)[\cos\delta \sin^2(\omega t) - \sin\delta \sin(\omega t)\cos(\omega t) + \omega t \sin\delta]\big|_a^b, \quad (8)$$

where the limits of integration will be chosen in different ways for particular cases. The energy dissipated in a complete cycle, ΔW, is determined by evaluating equation (8) over the limits 0 and $2\pi/\omega$,

$$\Delta W = \pi\sigma_0\gamma_0 \sin\delta. \quad (9)$$

This result has been reported many times [1–9]. The shaded areas labeled W^+ (above the time axis) are the work done *on* the material (Figure 1). Some of this work is stored in the material and some is dissipated in the form of heat. The shaded areas labelled W^-

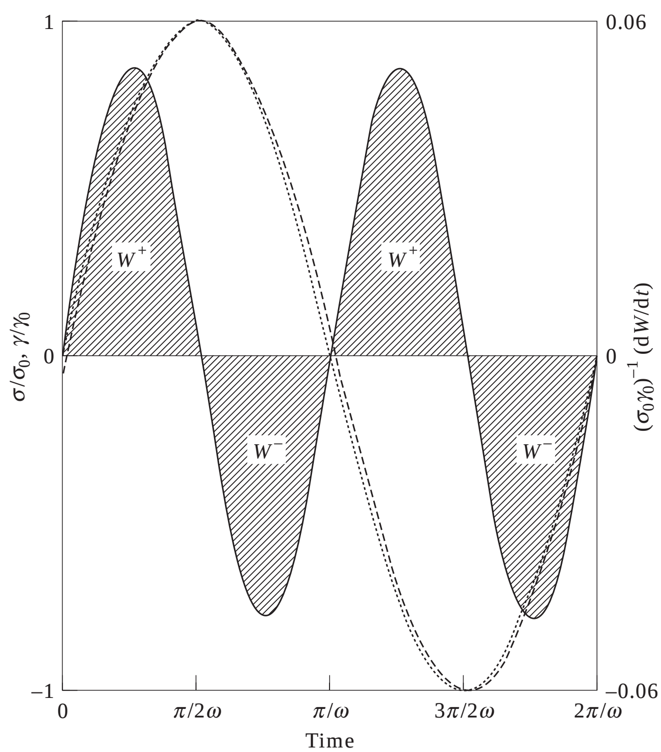

Figure 1. Stress, strain, and instantaneous work as a function of reduced time for $\delta = 0{\cdot}05$: · · · · , σ/σ_0; ----, γ/γ_0; ——, $(\sigma_0\gamma_0)^{-1}$ (dW/dt).

(below the time axis) are the work done *by* the material and therefore represents the stored energy. The difference between the work done (W^+) and the work stored (W^-) is the work dissipated. In the limit of a perfectly elastic material ($\delta = 0$), all of the work done on the material is stored and returned, therefore $2W^+ = 2W^-$. In the limit of a perfect viscous material ($\delta = \pi/2$), all the work is dissipated as heat and $\Delta W = \pi\sigma_0\gamma_0$. Under this condition, there is no work stored ($W^- = 0$), but only work done.

Note that the usual evaluation of work stored is approximated by integrating the first quarter cycle. This integration approximates the work done not the work stored, as shown in Figure 1. The two values are very close when the loss angle is small but differ significantly for high loss materials.

The maximum stored energy is evaluated in the following manner. Following the usual interpretation of maximum stored energy to include only the compression phase and not the tension phase, the quantity to be determined is W^- in the first half cycle only. Thus the limits to be used in equation (8) are $(\pi/2 + \delta)/\omega$ and (π/ω), and the maximum stored energy is

$$W_s = (\sigma_0\gamma_0/2)[(\pi/2 - \delta)\sin\delta - \cos\delta]. \tag{10}$$

From this result, it follows that the specific damping capacity for arbitrary δ is

$$\Delta W/W_s = 2\pi\tan\delta/[(\pi/2 - \delta)\tan\delta - 1]. \tag{11}$$

This equation is valid for arbitrary δ. For $\delta \ll 1$, equation (11) reduces to equation (1) except for the sign. Equation (11) yields a negative value for specific damping capacity because work stored is negative work. Equation (1) was stated to be for work stored but it was work done that was used and work done is positive work. In the limit as δ approaches 0, $\Delta W/W_s$ is zero and in the limit as δ approaches $\pi/2$, $\Delta W/W_s$ increases to infinity.

Having the exact relation between specific damping capacity and δ, one can now determine the range of validity of the small δ approximation. For $\delta = 0{\cdot}01$, the error in approximating equation (11) by equation (1) is 1·6%. Typical loss angle measurements have this much uncertainty, therefore a value of $\delta = 0{\cdot}01$ is a reasonable upper limit for the validity of equation (1).

To illustrate of the effect of high loss material on the work, the equations used in Figure 1 are replotted for $\delta = 0{\cdot}5$ in Figure 2. The area of the stored energy (W^-) is much smaller than the work done (W^+). Hence, in addition to the sign difference, there is a significant error in equating work done with work stored. The results in Figure 2 also illustrate the error in assuming that the maximum work done occurs at a quarter cycle $(\pi/2\omega)$ rather than $(\pi/2 + \delta)/\omega$. Even if the correct limit was used, only half of the energy stored per cycle is accounted for by equation (11).

3. NEW DEFINITION FOR SPECIFIC DAMPING CAPACITY

There are two drawbacks of the usual definition of specific damping capacity. The definition of ΔW is straightforward: it is the energy absorbed in one cycle. However, some difficulties arise with W_s, especially when considering high loss materials. First, the usual evaluation of the work stored is in fact not the work stored but the work done. This leads to significant error for high loss materials. Second, it appears inconsistent that ΔW is evaluated for a complete cycle whereas W_s is evaluated only in a quarter cycle. Qualitatively, this corresponds to considering the stored energy in the compression phase of the cycle and neglecting the energy stored in the tension phase. This omission leads to a factor of two since the energy stored in compression is equal to the energy stored in

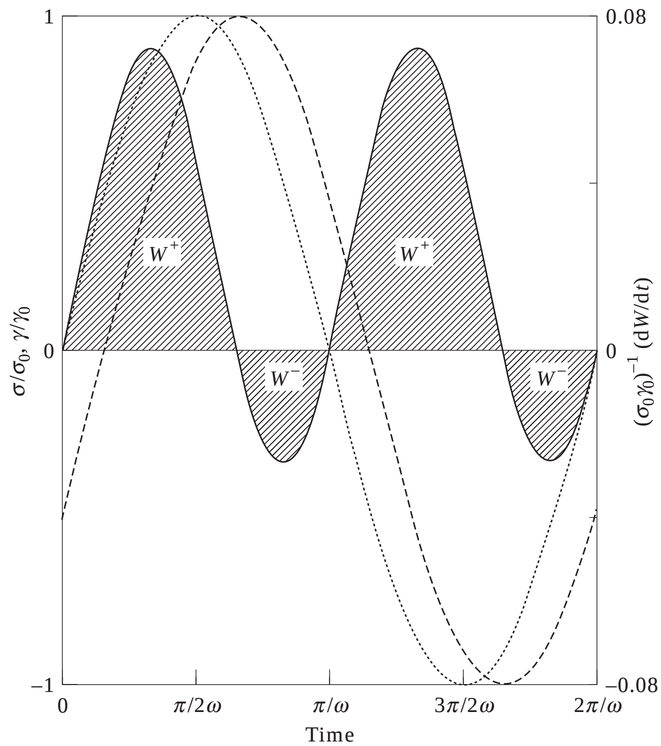

Figure 2. Stress, strain, and instantaneous work as a function of reduced time for $\delta = 0{\cdot}5$: $\cdots\cdots$, σ/σ_0; ----, γ/γ_0; ——, $(\sigma_0\gamma_0)^{-1}$ (dW/dt).

tension. Third, the form "$\Delta W/W$" implies a fractional quantity, less than one. But since stored energy can approach zero, as in a liquid, the specific damping capacity can be greater than one and in fact can approach infinity. This definition can lead to some confusion, though admittedly if one follows the definition correctly, there is no problem. To make the definition agree with what is intuitively assumed about specific damping capacity, a more logical definition would be to use W_d in the definition rather than W_s. Then the definition of specific damping capacity would be the work absorbed per cycle divided by the work done per cycle. This definition represents the fraction of the work done that is converted to heat.

Using the analysis already presented, the new definition can be evaluated easily. W_d is the total work done per cycle and consists of W^+ in the first half cycle plus W^+ in the second half cycle. W^+ for the first half cycle is determined by evaluating equation (8) from 0 to $(\pi/2 + \delta)/\omega$ which yields

$$W^+ = (\sigma_0\gamma_0/2)[\cos\delta + (\pi/2 + \delta)\sin\delta]. \tag{12}$$

W^+ for the second half cycle is determined by evaluating equation (8), again, using limits from π/ω to $(3\pi/2 + \delta)/\omega$ which yields an equation identical to equation (12). The two areas are equal, so that $W_d = 2W^+$. From this it follows that the new definition for specific damping capacity for arbitrary δ yields the relation

$$\Delta W/W_d = \pi\tan\delta/(1 + (\pi/2 + \delta)\tan\delta). \tag{13}$$

For $\delta \ll 1$, equation (13) reduces to

$$\Delta W/W_d \approx \pi \tan \delta, \qquad \delta \ll 1, \tag{14}$$

which is a factor two less than equation (1). This difference is due to defining W_d for a complete cycle rather than for a quarter cycle. In the limit as δ approaches 0, $\Delta W/W_d$ is zero and in the limit as δ approaches $\pi/2$, $\Delta W/W_d$ is 1, which is consistent with the new definition of specific damping capacity as the fraction of the total energy converted to heat. The upper limit for the validity of equation (14), approximating equation (13), is $\delta = 0{\cdot}01$, the same as for the earlier definition of specific damping capacity.

A plot comparing the two approximations of the usual definition of specific damping capacity (equations (1) and (2)) with the exact value of the usual definition (equation (11)) and the exact value of the new definition (equation (13)) as functions of δ is shown in Figure 3. The approximations are plotted beyond their range of validity ($\delta < 0{\cdot}01$) but they are sometimes used in this range, partly because the range of validity has not previously been determined. As can be seen, for δ greater than 0·16, equations (1) and (2) are greater than one. Equation (13) asymptotically approaches one as δ increases to infinity.

4. HYSTERESIS

The specific damping capacity has sometimes been determined from stress–strain hysteresis measurements rather than from time domain measurements. The analysis carried out here can be readily translated to hysteresis measurements.

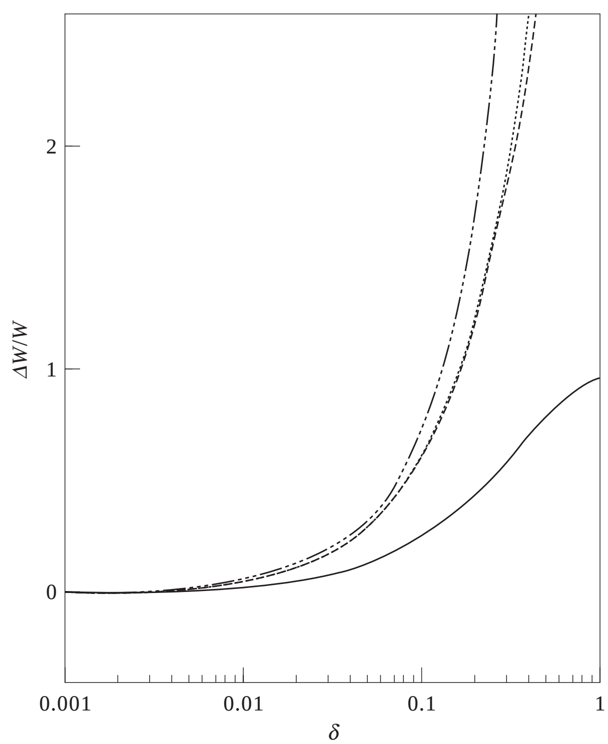

Figure 3. Specific damping capacity as a function of loss angle: · · · · , equation (1); ----, equation (2); — · · —, equation (11), ——, equation (13).

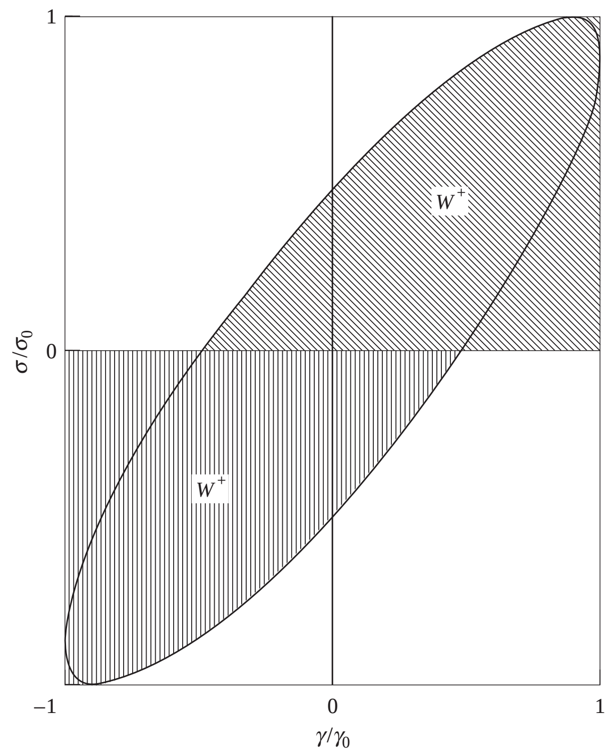

Figure 4. Stress–strain hysteresis loop for $\delta = 0{\cdot}5$.

The hysteresis loop is analyzed using equation (13) to determine $\Delta W/W_d$. Since it is well known that ΔW is the area within the hysteresis loop [8], only W_d is needed to determine the specific damping capacity. A hysteresis loop is obtained by plotting the stress (equation (3)) versus strain (equation (4)) as shown in Figure 4 for $\delta = 0{\cdot}5$. As was shown above, W_d is the sum of the two W^+ areas. By relating the two W^+ areas from the time domain results of Figure 2 to the hysteresis plot of Figure 4, W_d is easily determined. The W^+ area in the first half cycle (Figure 2) begins when the stress is zero ($t = 0$) and ends when the reduced strain (γ/γ_0) is one. Applying these conditions to Figure 4, the first W^+ area is clearly the shaded area above the strain axes. The W^+ in the second half cycle (Figure 2) begins when the stress is zero ($t = \pi/2$) and ends when the reduced strain is negative one. The second W^+ (Figure 4) is the shaded area below the strain axes. The two areas were numerically integrated and found to be equal to the results of equation (13) (analytical solution to the area in the time domain) evaluated at $\delta = 0{\cdot}5$. By analyzing the hysteresis loop in the above manner, the specific damping capacity will have a range from 0 to 1, as expected.

5. CONCLUSIONS

Based on the analysis reported here, the following conclusions were reached:

(1) The usual relation of specific damping capacity and loss angle is only valid for $\delta < 0{\cdot}01$.

(2) For arbitrary loss angle, specific damping capacity defined for work stored in the first half cycle is $\Delta W/W_s = 2\pi \tan\delta/[(\pi/2 - \delta)\tan\delta - 1]$.

(3) For arbitrary loss angle, specific damping capacity defined for work done per cycle is $\Delta W/W_d = \pi \tan\delta/[1 + (\pi/2 + \delta)\tan\delta]$, which has the desirable property of taking values between zero and one.

(4) The results obtained for time domain measurements can be directly transferred to stress-strain hysteresis measurements.

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